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Inequalities for Generalized Hypergeometric Functions*

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It is shown that some well-known Padé approximations for a particular form of the Gaussian hypergeometric function and two of its confluent forms give upper and lower bounds for these functions under suitable restrictions on the parameters and variable. With the aid of the beta and Laplace transforms, two-sided inequalities are derived for the generalized hypergeometric function ${}_pF_q$, $p = q$ or $p = q + 1$, and for a particular form of Meijer's G-function. Several examples are developed. These include upper and lower bounds for certain elementary functions, complete elliptic integrals, the incomplete gamma function, modified Bessel functions, and parabolic cylinder functions.

1. BASIC EQUALITIES

In this section, we give certain definitions and formulas needed to derive our main results. The notation used in [1] is followed throughout. We also make rather free use of results given in these volumes.

The generalized hypergeometric series is formally defined as

$$ {}_pF_q\left(\begin{matrix}\alpha_1, \alpha_2, \ldots, \alpha_p \\ \rho_1, \rho_2, \ldots, \rho_q\end{matrix}\,\middle|\, z\right) = \sum_{k=0}^{\infty} \left(z^k \prod_{j=1}^{p} (\alpha_j)_k \middle/ k! \prod_{j=1}^{q} (\rho_j)_k\right), \tag{1.1} $$

where

$$ (\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1) = \Gamma(\alpha+k)/\Gamma(\alpha). \tag{1.2} $$

It is convenient to employ a shorthand notation and write (1.1) in the form

$$ {}_pF_q(\alpha_p\,;\rho_q\,;z) = {}_pF_q\left(\begin{matrix}\alpha_p \\ \rho_q\end{matrix}\,\middle|\, z\right) = \sum_{k=0}^{\infty} (\alpha_p)_k\, z^k/[(\rho_q)_k k!]. \tag{1.3} $$

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In general, $\Gamma(\alpha_p + k)$ is interpreted as

$$\prod_{j=1}^{p} \Gamma(\alpha_j + k); \qquad (\alpha_p)_k \quad \text{as} \quad \prod_{j=1}^{p} (\alpha_j)_k ; \qquad (\alpha_p + \lambda) \quad \text{as} \quad \prod_{j=1}^{p} (\alpha_j + \lambda); \text{ etc.}$$

An empty term is treated as unity. The α_j's and ρ_j's are called numerator and denominator parameters, respectively, and z is called the variable. Where no confusion can ensue, we simply refer to (1.3) as a ${}_pF_q$.

We suppose throughout the entire paper that no denominator parameter is a negative integer or zero. The series (1.1) converges for all z if $p \leqslant q$. It diverges for all z, $z \neq 0$, if $p > q + 1$ unless one of the numerator parameters is a negative integer in which event (1.1) is a polynomial. If $p = q + 1$, (1.1) is absolutely convergent for $|z| < 1$. Let

$$\eta = \sum_{j=1}^{q+1} \alpha_j - \sum_{j=1}^{q} \rho_j . \tag{1.4}$$

Then the series (1.1) with $p = q + 1$ is

$$\begin{aligned} &\text{absolutely convergent for } |z| = 1 \quad \text{if } \mathrm{Re}(\eta) < 0, \\ &\text{conditionally convergent for } |z| = 1, \quad z \neq 1, \quad \text{if } 0 \leqslant \mathrm{Re}(\eta) < 1, \\ &\text{divergent for } |z| = 1 \quad \text{if } \mathrm{Re}(\eta) \geqslant 1. \end{aligned} \tag{1.5}$$

If $p = q + 1$, the series (1.1) can be analytically continued into the cut plane $|\arg(1 - z)| < \pi$, and in this case we use the same notation for the analytically continued function as for the series.

We shall need the following integral representations [2]:

$$\begin{aligned} {}_2F_1(\alpha, \beta; \gamma; -z) &= \frac{\Gamma(\gamma)}{\Gamma(\gamma - \alpha)\, \Gamma(\alpha)} \int_0^1 \frac{t^{\alpha-1}(1 - t)^{\gamma-\alpha-1}}{(1 + zt)^{\beta}}\, dt, \\ \mathrm{Re}(\gamma) > \mathrm{Re}(\alpha) > 0, &\qquad |\arg(1 + z)| < \pi; \end{aligned} \tag{1.6}$$

$${}_{p+1}F_{q+1}\left(\begin{matrix}\delta, \alpha_p \\ \epsilon, \rho_q\end{matrix}\,\middle|\, -z\right) = \frac{\Gamma(\epsilon)}{\Gamma(\delta)\, \Gamma(\epsilon - \delta)} \int_0^1 t^{\delta-1}(1 - t)^{\epsilon-\delta-1}\, {}_pF_q\left(\begin{matrix}\alpha_p \\ \rho_q\end{matrix}\,\middle|\, -zt\right) dt,$$

$$\mathrm{Re}(\epsilon) > \mathrm{Re}(\delta) > 0, \quad p \leqslant q; \qquad \text{or} \quad p = q + 1 \quad \text{and} \quad |\arg(1 + z)| < \pi. \tag{1.7}$$

Equality (1.6) is a special case of (1.7) since

$${}_1F_0(\sigma; -z) = (1 + z)^{-\sigma}, \qquad |\arg(1 + z)| < \pi. \tag{1.8}$$

Also (1.7) gives a well-known integral representation for ${}_1F_1$ since ${}_0F_0(-z) = e^{-z}$. Formulas like (1.6) and (1.7) are known as beta transforms. We also need the Laplace transforms

$$\int_0^\infty e^{-zt} t^{\epsilon-1} {}_pF_q\left(\begin{matrix}\alpha_p \\ \rho_q\end{matrix} \,\middle|\, \omega t\right) dt = \Gamma(\epsilon)\, z^{-\epsilon} {}_{p+1}F_q\left(\begin{matrix}\epsilon, \alpha_p \\ \rho_q\end{matrix} \,\middle|\, \omega/z\right),$$

$$p < q, \quad |\arg z| < \pi/2; \qquad p = q, \quad |\arg z| < \pi/2, \quad |\arg(z-\omega)| < \pi/2; \tag{1.9}$$

$$\int_0^\infty e^{-zt} t^{\epsilon-1} {}_pF_{p-1}\left(\begin{matrix}\alpha_p \\ \rho_{p-1}\end{matrix} \,\middle|\, -yt\right) dt = \frac{\Gamma(\rho_{p-1})\, z^{-\epsilon}}{\Gamma(\alpha_p)} G_{p,p+1}^{p+1,1}\left(z/y \,\middle|\, \begin{matrix}1, \rho_{p-1} \\ \epsilon, \ \alpha_p\end{matrix}\right),$$

$$\mathrm{Re}(\epsilon) > 0, \qquad |\arg z| < \pi/2, \qquad |\arg y| < \pi, \tag{1.10}$$

where $G_{p,q}^{m,n}(z)$, a generalization of ${}_pF_q(z)$, is Meijer's G-function; see [3]. Equations (1.9) and (1.10) hold also under some other conditions. A complete description of conditions is given in the reference cited.

For the present study, we record the expansion formula

$$G_{p,p+1}^{p+1,1}\left(z \,\middle|\, \begin{matrix}a_p \\ b_{p+1}\end{matrix}\right) = \sum_{h=1}^{p+1} \left(\Gamma(b_j - b_h)^* \,\Gamma(1 + b_h - a_1)\, z^{b_h} \Big/ \prod_{j=2}^{p} \Gamma(a_j - b_h)\right)$$

$$\times {}_pF_p\left(\begin{matrix}1 + b_h - a_p \\ 1 + b_h - b_{p+1}^*\end{matrix} \,\middle|\, z\right), \tag{1.11}$$

where the asterisk (*) sign means that the terms involving $b_j - b_h$ are to be omitted when $h = j$. We also have the asymptotic expansion

$$G_{p,p+1}^{p+1,1}\left(z \,\middle|\, \begin{matrix}a_p \\ b_{p+1}\end{matrix}\right) \sim \Gamma(1 + b_{p+1} - a_1)\, z^{a_1-1} {}_{p+1}F_{p-1}\left(\begin{matrix}1 + b_{p+1} - a_1 \\ 1 + a_p - a_1{}^*\end{matrix} \,\middle|\, -1/z\right),$$

$$|z| \to \infty, \qquad |\arg z| \leqslant 3\pi/2 - \delta, \qquad \delta > 0; \tag{1.12}$$

the asterisk (*) sign means that the term involving $1 + a_h - a_1$ is to be omitted when $h = 1$. A useful result for the G-function is

$$z^\sigma G_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix}a_p \\ b_q\end{matrix}\right) = G_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix}a_p + \sigma \\ b_p + \sigma\end{matrix}\right). \tag{1.13}$$

The special case $p = 1$ of (1.11) gives the confluent hypergeometric function

$$\psi(a; c; z) = [z^a \Gamma(a)\, \Gamma(a + 1 - c)]^{-1} G_{1,2}^{2,1}\left(z \,\middle|\, \begin{matrix}1 \\ a, 1 + a - c\end{matrix}\right) \tag{1.14}$$

$$= \frac{\Gamma(1-c)}{\Gamma(1 + a - c)} w_1 + \frac{\Gamma(c-1)}{\Gamma(a)} w_2,$$

$$w_1 = {}_1F_1(a; c; z), \qquad w_2 = z^{1-c} {}_1F_1(1 + a - c, 2 - c; z). \tag{1.15}$$

For ${}_2F_1$ and ${}_1F_1$, the following Kummer's transformation formulas are useful for analytic continuation and extension of inequalities for these functions.

$${}_2F_1(\alpha, \beta; \gamma; z) = (1-z)^{\gamma-\alpha-\beta}{}_2F_1(\gamma-\alpha, \gamma-\beta; \gamma; z), \tag{1.16}$$

$${}_2F_1(\alpha, \beta; \gamma; z) = (1-z)^{-\alpha}{}_2F_1\left(\alpha, \gamma-\beta; \gamma; -\frac{z}{1-z}\right), \tag{1.17}$$

$${}_1F_1(a; c; z) = e^z{}_1F_1(c-a; c; -z). \tag{1.18}$$

Also

$$\psi(a; c; z) = z^{1-c}\psi(1+a-c; 2-c; z), \tag{1.19}$$

which follows from (1.13).

The building blocks for inequalities for the ${}_pF_q$ and related functions are certain Padé approximations and inequalities for the Gaussian hypergeometric function ${}_2F_1$, one of whose numerator parameters is unity, and certain Padé approximations and inequalities for two forms of the incomplete gamma function.

We conclude this section with the definition of the Padé matrix table, and then, in the next section, we present the approximations and inequalities noted above.

Let

$$E(z) = \sum_{k=0}^{\infty} a_k z^k, \qquad |z| < r, \tag{1.20}$$

be approximated by

$$E_{p,q}(z) = A_p(z)/B_q(z), \tag{1.21}$$

where $A_p(z)$ and $B_q(z)$ are polynomials in z of degree p and q, respectively. If

$$B_q(z)\, E(z) - A_p(z) = z^{p+q+1} H_{p,q}(z), \qquad H_{p,q}(0) \neq 0, \tag{1.22}$$

then $E_{p,q}(z)$ is that Padé approximation of $E(z)$ which occupies the position (p, q) of the Padé matrix table. If $p = q$, we have a main diagonal Padé approximation. The definition carries through in a formal sense if the series in (1.20) is divergent, but asymptotic to $E(z)$ in some sector of the complex plane.

II. Padé Approximations and Inequalities for a Particular Gaussian Hypergeometric Function

THEOREM 1. *Let*

$$E(z) = {}_2F_1(1, \sigma; \rho + 1; -z), \tag{2.1}$$

$$E_n(z, a) = \varphi_n(z)/f_n(z),$$

$$f_n(z) = {}_2F_1(-n, n + \rho + 1 - a; \sigma + 1 - a; -1/z), \quad a = 0 \text{ or } 1,$$

$$\varphi_n(z) = [n(n + \rho + 1 - a)/\sigma z)]^a \sum_{k=0}^{n-a} \frac{(-)^k (a - n)_k (n + \rho + 1)_k z^{-k}}{(\sigma + 1)_k (a + 1)_k}$$

$$\times\, {}_4F_3\left(\begin{matrix} -n + a + k, n + \rho + 1 + k, \sigma, 1 \\ \sigma + 1 + k, a + 1 + k, \rho + 1 \end{matrix} \,\middle|\, 1\right). \tag{2.2}$$

$$R_n(z) = F_n(z)/f_n(z),$$

$$F_n(z) = \frac{(-\sigma)^{1-a} \rho^a (\rho + 1 - \sigma)_n (z + 1)^{\rho - \sigma}}{(\rho + 1 - a)_n z^{n+\rho}} \int_0^z \frac{(z - t)^n t^{n+\rho-a}}{(t + 1)^{n+\rho+1-\sigma}}\, dt,$$

$$\mathrm{Re}(\rho) > a - 1 - n. \tag{2.3}$$

Then

$$E(z) = E_n(z, a) + R_n(z); \tag{2.4}$$

the approximations $E_n(z, a)$ *occupy the positions* $(n - a, n)$ *of the Padé matrix table, and, if* z, σ *and* ρ *are fixed,* $z \neq -1$, $|\arg(1 + z)| < \pi$, *then*

$$\lim_{n \to \infty} R_n(z) = 0. \tag{2.5}$$

For proof and many other details concerning (2.1)–(2.5) including effective asymptotic estimates of $R_n(z)$, see [4]. On p. 170 of this reference, the function treated is the above $E(z)$ with z replaced by $1/z$.

We now establish the following

LEMMA. *Let*

$$z > 0, \qquad \sigma + 1 - a > 0, \qquad n + \rho + 1 - a > 0.$$

Then

$$\operatorname{sgn} R_n(z) = \operatorname{sgn} \left\{ \frac{(-\sigma)^{1-a} \rho^a (\rho + 1 - \sigma)_n}{(\rho + 1 - a)_n} \right\}. \tag{2.6}$$

Proof. Under the stated hypotheses, $f_n(z)$ is a series of positive terms and so is positive. The integral portion of $F_n(z)$ is also positive and (2.6) follows at once.

The following result is an immediate consequence of the lemma and is of prime importance to our studies.

THEOREM 2. *If $z > 0$, $\rho \geqslant 0$, $\sigma > 0$, $\rho + 1 - \sigma > 0$, then*

$$E_n(z, 1) < E(z) < E_m(z, 0), \qquad m, n > 0. \tag{2.7}$$

Further, if $z > 0$, $\rho \geqslant 0$, $\sigma > 0$, $\rho + 1 - \sigma < 0$, and $\rho + 1 - \sigma$ is not a negative integer or zero, then (2.7) *holds provided $(\rho + 1 - \sigma)_r$ is positive, $r = n$ or $r = m$; but if $(\rho + 1 - \sigma)_r$ is negative, then* (2.7) *holds with reversed inequality signs.*

If $z = 0$ or if $\sigma = 0$, the inequalities become equalities. In general, throughout our work, inequalities for ${}_pF_q(\alpha_p; \rho_q; z)$ become equalities if $z = 0$ or if any numerator parameter is zero. In the sequel, we usually omit such statements.

Further inequalities for other choices of the parameters ρ, σ, and a can be readily deduced from (2.4) and (2.6). We omit details. Additional inequalities can be obtained, when either σ or ρ or both are less than -1 but neither is a negative integer, from the general result that, if r is a positive integer or zero,

$${}_pF_q\left(\begin{matrix}\alpha_p \\ \rho_q\end{matrix}\,\middle|\, z\right) = \sum_{k=0}^{r-1} \frac{(\alpha_p)_k z^k}{(\rho_q)_k k!} + \frac{(\alpha_p)_r z^r}{(\rho_q)_r r!}\, {}_{p+1}F_{q+1}\left(\begin{matrix}\alpha_p + r, 1 \\ \rho_q + r, r + 1\end{matrix}\,\middle|\, z\right). \tag{2.8}$$

For if any numerator parameter in ${}_pF_q$ on the left is one, the ${}_{p+1}F_{q+1}$ on the right becomes a ${}_pF_q$ and also has a numerator parameter which is one. Thus, in the case that $p = 2$, $q = 1$, and $\alpha_2 = 1$, we can employ inequalities for the ${}_2F_1$ on the right of (2.8) to get inequalities for the ${}_2F_1$ on the left of (2.8). As will be seen, (2.8) is useful in extending the domain of validity of general inequalities for the ${}_pF_q$.

As previously remarked, (1.16) and (1.17) are useful in extending inequalities for the ${}_2F_1$. The z-range of validity of inequalities for general ${}_2F_1$'s can always be extended by use of other well-known formulas for analytic continuation, and the range on the parameters can be extended by use of contiguous relations. For a complete discussion of analytic continuation and contiguous relations for ${}_1F_1$'s, ${}_2F_1$'s, and ${}_pF_q$'s, see [5].

The special case $\rho = 0$ of (2.1)–(2.4) is important for applications. In this instance,

$$E(z) = (1+z)^{-\sigma} = E_n(z, a) + R_n(z), \tag{2.9}$$

where $E_n(z, a)$ and $R_n(z)$ have the same meaning as before, and

$$f_n(z) = {}_2F_1(-n, n+1-a; \sigma+1-a; -1/z),$$

$$\varphi_n(z) = \left(\frac{nz^{-1}}{1-\sigma}\right)^{-a} \frac{(1-\sigma)_n}{(\sigma+1-a)_n} {}_2F_1(a-n, n+1; 1+a-\sigma; -1/z), \tag{2.10}$$

$$F_n(z) = \frac{(-\sigma)^{1-a}(1-\sigma)_n(z+1)^{-\sigma}}{(n-a)!\,z^n} \int_0^z \frac{(z-t)^n t^{n-a}}{(t+1)^{n+1-\sigma}}\,dt, \qquad n \geqslant a. \tag{2.11}$$

Thus from the first statement of (2.7), we can deduce

THEOREM 3. *If* $z > 0$, $0 < \sigma < 1$, *and* r *is an integer, then, with* $E_n(z, a) = \varphi_n(z)/f_n(z)$ *as in* (2.10),

$$(1+z)^r E_n(z, 1) < (1+z)^{r-\sigma} < (1+z)^r E_m(z, 0), \qquad m, n, > 0, \tag{2.12}$$

$$\frac{(1+z)^r}{E_m(z, 0)} < (1+z)^{r+\sigma} < \frac{(1+z)^r}{E_n(z, 1)}, \qquad m, n > 0. \tag{2.13}$$

Another inequality for $\sigma > 1$ follows from the second statement of Theorem 2. Further inequalities follow from (2.6) and (2.9), and from the discussion surrounding (2.8). If $\sigma = 1/2$, the polynomials $f_n(z)$ and $\varphi_n(z)$ are related to Chebyshev polynomials of the first and second kinds, respectively. For details, see [6].

III. PADÉ APPROXIMATIONS AND INEQUALITIES FOR INCOMPLETE GAMMA FUNCTIONS

There are two forms of the incomplete gamma function. First, we have

$$\begin{aligned} H(\nu, z) &= \nu z^{-\nu} e^{-z-i\nu\pi} \gamma(\nu, z e^{i\pi}) \\ &= \nu z^{-\nu} e^{-z} \int_0^z e^t t^{\nu-1}\,dt, \qquad \operatorname{Re}(\nu) > 0 &(3.1) \\ &= {}_1F_1(1; \nu+1; -z). &(3.2) \end{aligned}$$

THEOREM 4. *Let*

$$H_n(\nu, z, a) = A_n(\nu, z)/B_n(\nu, z),$$

$$B_n(\nu, z) = {}_1F_1(-n; -2n + a - \nu; z)$$

$$= \frac{z^n}{(n+\nu+1-a)_n}\, {}_2F_0(-n, n+\nu+1-a; -1/z), \quad a = 0 \text{ or } 1,$$

$$A_n(\nu, z) = \left[\frac{n(n+\nu)}{z}\right]^a \frac{z^n}{(n+\nu+1-a)_n} \sum_{k=0}^{n-a} \frac{(a-n)_k(n+\nu+1)_k}{(\nu+1)_k(1+a)_k}$$

$$\times\, {}_3F_1\left(\begin{matrix}-n+a+k, n+\nu+1+k, 1\\ 1+a+k\end{matrix}\,\middle|\, -1/z\right), \tag{3.3}$$

$$V_n(\nu, z) = P_n(\nu, z)/B_n(\nu, z),$$

$$P_n(\nu, z) = \frac{(-1)^{n+1-a}\Gamma(\nu+1)\, z^{-\nu}e^{-z}}{\Gamma(2n+\nu+1-a)} \int_0^z (z-t)^n e^{\nu t} t^{n+\nu-a}\, dt,$$

$$\mathrm{Re}(\nu) > a - 1 - n. \tag{3.4}$$

Then

$$H(\nu, z) = H_n(\nu, z, a) + V_n(\nu, z), \tag{3.5}$$

the approximations $H_n(\nu, z, a)$ occupy the positions $(n - a, n)$ of the Padé matrix table, and, if z and ν are fixed,

$$\lim_{n\to\infty} V_n(\nu, z) = 0. \tag{3.6}$$

For the proof and other developments concerning (3.2)–(3.6) including an efficient asymptotic estimate for $V_n(\nu, z)$, see [7]. Except for certain normalization factors introduced in $A_n(\nu, z)$ and $B_n(\nu, z)$, (3.2)–(3.5) follow from (2.1)–(2.4) by confluence. That is, in the latter equalities, put $\rho = \nu$, replace z by z/σ and let $\sigma \to \infty$. The analog of (2.7) is

THEOREM 5. *If $z > 0$ and $\nu > -1$, then*

$$H_n(\nu, z, 1) >(<)\, H(\nu, z) >(<)\, H_m(\nu, z, 0), \qquad m > n > 0 \tag{3.7}$$

where the $>(<)$ sign pertains if both m and n are odd (even). Further, if $z > 0$, $\nu < 0$, but ν is not a negative integer, $m + \nu + 1 - a > 0$, $n + \nu + 1 - a > 0$, and $r < -\nu < r + 1$, where r is a positive integer or zero, then (3.7) holds where the $>(<)$ sign pertains if both $r + n$ and $r + m$ are odd (even).

The proof is much akin to that of Theorem 2 and is omitted.

If $\nu \geqslant 0$ and $z = -x$, $x > 0$, then from (3.4), $P_n(\nu, -x)$ is positive (negative) if n is even (odd). Thus additional inequalities can be written once the sign of $B_n(\nu, -x)$ is determined. It is known (see [8]) that if ν and x are fixed and restricted as above, $B_n(\nu, -x)$ is positive provided n is sufficiently large.

If ν is negative but not an integer, further inequalities follow from (3.7) and the relation

$$H(\nu, z) = \sum_{k=0}^{s-1} \frac{(-)^k z^k}{(\nu+1)_k} + \frac{(-)^s z^s}{(\nu+1)_s} H(\nu + s, z). \tag{3.8}$$

Next we consider the complementary incomplete gamma function

$$\Gamma(\nu, z) = \int_z^\infty t^{\nu-1} e^{-t}\, dt = \Gamma(\nu) - \gamma(\nu, z), \qquad \mathrm{Re}(z) > 0. \tag{3.9}$$

We have the further integral representations

$$\Gamma(\nu, z) = z^\nu e^{-z} \int_0^\infty e^{-zt}(1+t)^{\nu-1}\, dt, \qquad \mathrm{Re}(z) > 0, \tag{3.10}$$

$$= \frac{e^{-z}}{\Gamma(1-\nu)} \int_0^\infty e^{-zt} t^{-\nu}(1+t)^{-1}\, dt, \quad \mathrm{Re}(z) > 0, \quad \mathrm{Re}(\nu) < 1. \tag{3.11}$$

Also

$$\Gamma(\nu, z) = z^\nu e^{-z} \psi(1; 1+\nu; z) = e^{-z}[z^{1-\nu}\Gamma(1-\nu)]^{-1} G_{1,2}^{2,1}\left(z \,\middle|\, \begin{matrix} 1 \\ 1, 1-\nu \end{matrix}\right). \tag{3.12}$$

THEOREM 6. *Let*

$$\left.\begin{aligned} P_n(\nu, z, a) &= E_n(\nu, z)/F_n(\nu, z), \\ F_n(\nu, z) &= {}_1F_1(-n; 2-a-\nu; -z), \qquad a = 0 \quad \text{or } 1, \\ E_n(\nu, z) &= \left(\frac{nz}{1-z}\right)^a \sum_{k=0}^{n-a} \frac{(a-n)_k (1-\nu)_k}{(2-\nu)_k (1+a)_k} \\ &\quad \times {}_2F_2\left(\begin{matrix} -n+a+k, 1 \\ 2-\nu+k, 1+a+k \end{matrix} \,\middle|\, -z\right), \end{aligned}\right\} \tag{3.13}$$

$$T_n(\nu, z) = S_n(\nu, z)/F_n(\nu, z),$$

$$S_n(\nu, z) = (\nu-1)^{1-a} z^{1-\nu} e^z \int_z^\infty (t-z)^n t^{\nu+a-2-n} e^{-t}\, dt,$$

$$z \neq 0, \qquad |\arg z| < \pi. \tag{3.14}$$

Then

$$z^{1-\nu} e^z \Gamma(\nu, z) = P_n(\nu, z, a) + T_n(\nu, z), \tag{3.15}$$

the approximations $P_n(\nu, z, a)$ occupy the positions $(n - a, n)$ of the Padé matrix table, and if ν is fixed, z bounded and bounded away from the origin and $|\arg z| < \pi$, then

$$\lim_{n\to\infty} T_n(\nu, z) = 0 \tag{3.16}$$

except possibly in the neighborhood of zeros of $F_n(\nu, z)$.

For proof and other properties associated with (3.9)–(3.16), see [9]. There we develop for ν fixed an asymptotic estimate of the error $T_n(\nu, z)$ valid, for n large, uniformly in z, z bounded away from the origin. Formally (3.12)–(3.15) follow (by confluence) from (2.1)–(2.4) if in the latter we set $\sigma = 1 - \nu$, replace z by $z(\rho + 1)$, let $\rho \to \infty$ and then replace z by $1/z$.

THEOREM 7. *If $z > 0$ then*

$$\begin{aligned} &P_n(\nu, z, 1) < z^{1-\nu}e^z\Gamma(\nu, z) < P_m(\nu, z, 0), \qquad m, n > 0, \quad \text{if } \nu < 1, \\ &P_n(\nu, z, 0) < z^{1-\nu}e^z\Gamma(\nu, z) \qquad \text{if} \quad 1 < \nu < 2, \end{aligned} \tag{3.17}$$

with equality as $z \to \infty$.

The proof is similar to that of Theorem 2, and we omit details. We can obtain further inequalities for $\nu > 1$ by use of the relation

$$\Gamma(\nu, z) = z^{\nu-1}e^{-z} \sum_{k=0}^{r-1} (-)^k(1 - \nu)_k z^{-k} + (1 - \nu)_r\, \Gamma(\nu - r, z). \tag{3.18}$$

The incomplete gamma function is a special case of the confluent hypergeometric function which in turn can be viewed as a special case of the Gaussian hypergeometric function. As in the case of ${}_2F_1$'s, the range of validity of inequalities for the general ${}_1F_1$ and $\psi(a; c; z)$ functions can be extended by use of Kummer's formulas, analytic continuation formulas (see for instance (1.15), (1.18) and (1.19)), and contiguous relations. See [10] for further details and numerous other properties of confluent hypergeometric functions.

IV. INEQUALITIES FOR THE GAUSSIAN HYPERGEOMETRIC FUNCTION AND THE ${}_{p+1}F_p$

To get inequalities for a general ${}_2F_1$ under suitable restrictions, we propose to combine (1.6) with (2.12) or (2.13), as appropriate, and (2.7). Then by repeated use of (1.7) and (2.7) we can get inequalities for a ${}_{p+1}F_p$. We shall not carry this process through in all generality as it becomes quite complicated. Later we introduce some simplifications, but first some general useful remarks.

It is clear from the above comments that we need to express $E_n(z, a)$ as given by (2.2) as a sum of partial fractions. In certain applications, we shall also want $[E_n(z, a)]^{-1}$ as given by (2.13) in such a form. Now except for a multiplicative factor independent of z, $f_n(-1/z)$ is the shifted Jacobi polynomial $R_n^{(\alpha,\beta)}(z)$ with $\alpha = \rho - \sigma$ and $\beta = \sigma - a$. If $\alpha > -1$, $\beta > -1$, this latter polynomial has simple zeros only and they lie in $0 < z < 1$, see [11]. Thus if $0 < \sigma < \rho + 1$, the zeros of $f_n(z)$ are simple and lie in $-\infty < z < -1$. We can write

$$E_n(z, a) = V_n + \sum_{k=1}^{n} \frac{V_{n,k}}{1 + z\mu_{n,k}},$$

$$f_n(-\xi_{n,k}) = 0, \qquad V_{n,k} = \mu_{n,k}\frac{\varphi_n(x)}{f_n'(x)}, \qquad x = -\xi_{n,k} = -1/\mu_{n,k},$$

$$V_n + \sum_{k=1}^{n} V_{n,k} = 1; \qquad V_1 = 0 \quad \text{and} \quad V_{1,1} = 1 \quad \text{if } a = 1. \tag{4.1}$$

Now combine (1.6) and (2.12) with $r = 0$, $\sigma = \beta$, $0 < \beta < 1$, and use (4.1) with the understanding that $\rho = 0$ and $\sigma = \beta$. Then

$$\begin{gathered} G_n(z, 1) < {}_2F_1(\alpha, \beta; \gamma; -z) < G_m(z, 0), \\ z > 0, \qquad \gamma > \alpha > 0, \qquad 0 < \beta \leqslant 1, \qquad m, n > 0, \end{gathered} \tag{4.2}$$

where

$$G_n(z, a) = V_n + \sum_{k=1}^{n} V_{n,k}\, {}_2F_1(1, \alpha; \gamma; -z\mu_{n,k}), \tag{4.3}$$

with equality if $z = 0$ or if $\beta = 0$ or $\beta = 1$. Next apply (2.7) to (4.2) to get inequalities for ${}_2F_1(\alpha, \beta; \gamma; -z)$ expressed as a sum of partial fractions. (Notice that the values of m and n in (2.7), and m and n in (4.2) are not necessarily related.) In the expressions so obtained, replace z by zt, multiply by $t^{\delta-1}(1-t)^{\epsilon-\delta-1}\,dt$, integrate with respect to t from 0 to 1 and use (1.6). We then get an inequality of the form (4.2) with ${}_2F_1(\alpha, \beta; \gamma; -z)$ replaced by ${}_3F_2(\alpha, \beta, \delta; \gamma, \epsilon; -z)$. Iteration of this process leads to inequalities for ${}_{p+1}F_p$. Indeed in this manner we can get excellent approximations for ${}_{p+1}F_p$ by taking m and n sufficiently large without restricting the parameters and variable to be real. Obviously such a scheme is quite complicated. If approximations for ${}_pF_q$ are of main interest, then the simple developments in [12] are very effective. To achieve rather sharp inequalities, it is sufficient to consider the case $m = n = 1$ only. We now turn our attention to this case.

From (2.12) with $r = 0$ and $m = n = 1$, followed by use of (1.6), we get

$$[1 + \beta z]^{-1} < (1 + z)^{-\beta} < \frac{1-\beta}{1+\beta} + \frac{2\beta}{1+\beta}\left[1 + \frac{(1+\beta)z}{2}\right]^{-1}, \qquad 0 < \beta < 1; \tag{4.4}$$

$$\begin{aligned} {}_2F_1\left(\begin{matrix}\alpha, 1\\ \gamma\end{matrix}\,\middle|\, -\beta z\right) &< {}_2F_1\left(\begin{matrix}\alpha, \beta\\ \gamma\end{matrix}\,\middle|\, -z\right) \\ &< \frac{1-\beta}{1+\beta} + \frac{2\beta}{1+\beta}\,{}_2F_1\left(\begin{matrix}\alpha, 1\\ \gamma\end{matrix}\,\middle|\, -\frac{(1+\beta)z}{2}\right), \end{aligned} \tag{4.5}$$

$$z > 0, \qquad 0 < \beta < 1, \qquad \gamma \geqslant \alpha > 0.$$

Here we have equalities when $\beta = 0$ and when $\beta = 1$. $\gamma = \alpha$ is permitted as in this event (4.5) becomes (4.4). Now from the first statement of Theorem 2, if $m = n = 1$,

$$\left[1 + \frac{\alpha z}{\gamma}\right]^{-1} < {}_2F_1\left(\begin{matrix}\alpha, 1\\ \gamma\end{matrix}\,\middle|\, -z\right) < \frac{\gamma - \alpha}{\gamma(\alpha+1)} + \frac{\alpha(\gamma+1)}{\gamma(\alpha+1)}\left[1 + \frac{(\alpha+1)z}{\gamma+1}\right]^{-1},$$
$$z > 0, \qquad \gamma \geqslant \alpha > 0. \tag{4.6}$$

Combining (4.5) and (4.6), we find

$$\begin{aligned} \left[1 + \frac{\alpha\beta z}{\gamma}\right]^{-1} &< {}_2F_1\left(\begin{matrix}\alpha, \beta\\ \gamma\end{matrix}\,\middle|\, -z\right) < 1 - \frac{2\alpha\beta(\gamma+1)}{\gamma(\alpha+1)(\beta+1)} \\ &+ \frac{2\alpha\beta(\gamma+1)}{\gamma(\alpha+1)(\beta+1)}\left[1 + \frac{(\alpha+1)(\beta+1)z}{2(\gamma+1)}\right]^{-1}, \end{aligned} \tag{4.7}$$
$$z > 0, \qquad 0 < \beta \leqslant 1, \qquad \gamma \geqslant \alpha > 0.$$

In (4.7), replace z by zt, multiply throughout by $t^{\delta-1}(1-t)^{\epsilon-\delta-1}\,dt$, integrate and use (1.7). Then

$$\begin{aligned} {}_2F_1\left(\begin{matrix}\delta, 1\\ \epsilon\end{matrix}\,\middle|\, -\frac{\alpha\beta z}{\gamma}\right) &< {}_3F_2\left(\begin{matrix}\alpha, \beta, \delta\\ \gamma, \epsilon\end{matrix}\,\middle|\, -z\right) < 1 - \frac{2\alpha\beta(\gamma+1)}{\gamma(\alpha+1)(\beta+1)} \\ &+ \frac{2\alpha\beta(\gamma+1)}{\gamma(\alpha+1)(\beta+1)}\,{}_2F_1\left(\begin{matrix}\delta, 1\\ \epsilon\end{matrix}\,\middle|\, -\frac{(\alpha+1)(\beta+1)z}{2(\gamma+1)}\right), \end{aligned} \tag{4.8}$$
$$z > 0, \qquad 0 < \beta \leqslant 1, \qquad \gamma \geqslant \alpha > 0, \qquad \epsilon \geqslant \delta > 0.$$

Next apply (4.6) to each ${}_2F_1$ in (4.8) and so obtain

$$\left.\begin{aligned} &[1+uz]^{-1} < {}_3F_2\left(\begin{matrix}\alpha, \beta, \delta\\ \gamma, \epsilon\end{matrix}\,\middle|\, -z\right) < 1 - \frac{u}{v} + \frac{u}{v}[1+vz]^{-1},\\ &u = \frac{\alpha\beta\delta}{\gamma\epsilon}, \qquad v = \frac{(\alpha+1)(\beta+1)(\delta+1)}{2(\gamma+1)(\epsilon+1)},\\ &z > 0, \qquad 0 < \beta \leqslant 1, \qquad \gamma \geqslant \alpha > 0, \qquad \epsilon \geqslant \delta > 0. \end{aligned}\right\} \tag{4.9}$$

By induction we can establish

THEOREM 8.

$$\begin{aligned} [1+\sigma\theta z]^{-1} &< {}_{p+1}F_p\left(\begin{matrix}\sigma, \alpha_p\\ \rho_p\end{matrix}\,\middle|\, -z\right)\\ &< 1 - \frac{2\sigma\theta}{(\sigma+1)\varphi} + \frac{2\sigma\theta}{(\sigma+1)\varphi}\left[1 + \frac{(\sigma+1)\varphi z}{2}\right]^{-1},\\ &z > 0, \qquad 0 < \sigma \leqslant 1, \qquad \rho_j \geqslant \alpha_j > 0, \qquad j = 1, 2,..., p; \end{aligned} \tag{4.10}$$

here and throughout this paper we use the shorthand notation

$$\theta = \frac{\alpha_p}{\rho_p}, \qquad \varphi = \frac{\alpha_p + 1}{\rho_p + 1}, \qquad \eta = \frac{\alpha_p + 2}{\rho_p + 2}. \tag{4.11}$$

Another general result can be found in a like manner by starting with an inequality for ${}_2F_1(1, a; c; -z)$ valid for $z > 0$ and $0 < c \leqslant a$ which follows from the second statement of Theorem 2; see (2.7). We have

THEOREM 9.

$$\begin{aligned} 1 - \frac{a(c+1)}{c(a+1)} + \frac{a(c+1)}{c(a+1)}\left[1 + \frac{(a+1)\theta z}{c+1}\right]^{-1} &< {}_{p+2}F_{p+1}\left(\begin{matrix}1, a, \alpha_p\\ c, \rho_p\end{matrix}\,\middle|\, -z\right)\\ < 1 - \frac{\theta}{\varphi} + \frac{\theta}{\varphi}\left[1 + \frac{a\varphi z}{c}\right]^{-1},&\\ z > 0, \qquad 0 < c \leqslant a, \qquad \rho_j \geqslant \alpha_j > 0, \qquad j = 1, 2,..., p.& \end{aligned} \tag{4.12}$$

Next we consider inequalities for ${}_2F_1(\alpha, \beta; \gamma; -z)$ and its natural generalization ${}_{p+1}F_p$, when $1 < \beta < 2$. Use (2.12) with $r = -1$ and $\beta = \sigma + 1$. Then

$$\begin{aligned} \frac{1}{2-\beta}\left\{\frac{1}{1+z} - \frac{\beta-1}{1+(\beta-1)z}\right\} < (1+z)^{-\beta} &< \frac{\beta}{(2-\beta)(1+z)}\\ &- \frac{2(\beta-1)}{(2-\beta)(1+\beta z/2)},\\ z > 0, \qquad 1 < \beta < 2.& \end{aligned} \tag{4.13}$$

Employ the beta transform technique to obtain

$$\frac{1}{(2-\beta)}\left\{{}_2F_1\left(\begin{matrix}\alpha, 1\\ \gamma\end{matrix}\,\middle|\,-z\right) - (\beta-1)\,{}_2F_1\left(\begin{matrix}\alpha, 1\\ \gamma\end{matrix}\,\middle|\,-(\beta-1)z\right)\right\} < {}_2F_1\left(\begin{matrix}\alpha, \beta\\ \gamma\end{matrix}\,\middle|\,-z\right)$$

$$< \frac{\beta}{2-\beta}\,{}_2F_1\left(\begin{matrix}\alpha, 1\\ \gamma\end{matrix}\,\middle|\,-z\right) - \frac{2(\beta-1)}{(2-\beta)}\,{}_2F_1\left(\begin{matrix}\alpha, 1\\ \gamma\end{matrix}\,\middle|\,-\frac{\beta z}{2}\right), \tag{4.14}$$

$$z > 0, \qquad 1 < \beta < 2, \qquad \gamma \geqslant \alpha > 0.$$

The latter can be coupled with (4.6) and the entire process can be iterated to derive

THEOREM 10.

$$(2-\sigma)^{-1}\,[1+\theta z]^{-1} - \frac{\sigma-1}{2-\sigma} + \frac{(\sigma-1)\theta}{(2-\sigma)\varphi}\,[1 - \{1+(\sigma-1)\varphi z\}^{-1}]$$

$$< {}_{p+1}F_p\left(\begin{matrix}\sigma, \alpha_p\\ \rho_p\end{matrix}\,\middle|\,-z\right)$$

$$< \frac{\sigma}{2-\sigma} - \frac{2(\sigma-1)}{2-\sigma}\left[1+\frac{\sigma\theta z}{2}\right]^{-1} - \frac{\sigma\theta}{(2-\sigma)\varphi}\,[1-\{1+\varphi z\}^{-1}],$$

$$z > 0, \qquad 1 \leqslant \sigma \leqslant 2, \qquad \rho_j \geqslant \alpha_j > 0, \qquad j = 1, 2, \ldots, p \tag{4.15}$$

In a similar fashion, starting with

$$1 - \frac{\sigma}{\sigma+1}\,[1-\{1+(\sigma+1)z\}^{-1}] < (1+z)^{-\sigma}$$

$$< 1 - \frac{\sigma z}{\sigma+2} - \frac{2\sigma(\sigma+1)}{(\sigma+2)^2}\left[1-\left\{1+\frac{(\sigma+2)z}{2}\right\}^{-1}\right] \tag{4.16}$$

$$z > 0, \qquad -1 < \sigma < 0,$$

we can derive

THEOREM 11.

$$1 - \frac{\sigma\theta}{(\sigma+1)\varphi}\,[1-\{1+(\sigma+1)\varphi z\}^{-1}] < {}_{p+1}F_p\left(\begin{matrix}\sigma, \alpha_p\\ \rho_p\end{matrix}\,\middle|\,-z\right)$$

$$< 1 - \frac{\sigma\theta z}{\sigma+2} - \frac{2\sigma(\sigma+1)}{(\sigma+2)^2}\left[1-\left\{1+\frac{(\sigma+2)\theta z}{2}\right\}^{-1}\right], \tag{4.17}$$

$$z > 0, \qquad -1 < \sigma < 0, \qquad \rho_j \geqslant \alpha_j > 0, \qquad j = 1, 2, \ldots, p.$$

Under the hypotheses of (4.17), we can get an alternative inequality by combining (2.8) with $r = 1$ and (4.10). Thus we have

THEOREM 12.

$$\begin{aligned} &1 - \frac{2\sigma\theta}{(\sigma+1)\varphi} + \frac{2\sigma\theta}{(\sigma+1)\varphi}\left[1 + \frac{(\sigma+1)\varphi z}{2}\right]^{-1} < {}_{p+1}F_p\left(\begin{matrix}\sigma, \alpha_p\\ \rho_p\end{matrix}\,\middle|\, -z\right)\\ &\quad < 1 - \sigma\theta z + \frac{3\sigma(\sigma+1)\theta\varphi z}{2(\sigma+2)\eta} - \frac{9\sigma(\sigma+1)\theta\varphi}{2(\sigma+2)^2\eta^2}\\ &\quad \times\left[1 - \left(1 + \frac{(\sigma+2)\eta z}{3}\right)^{-1}\right],\\ &\quad z > 0, \qquad -1 < \sigma < 0, \qquad \rho_j \geqslant \alpha_j > 0, \qquad j = 1, 2,\ldots, p. \end{aligned} \tag{4.18}$$

It is of interest to compare (4.17) with (4.18). The left-hand side of (4.17) is less than the left-hand side of (4.18). The right-hand side of (4.18) is less than the right-hand side of (4.17) provided

$$\theta + \frac{(\sigma+2)\eta\theta z}{3} < \varphi + \frac{(\sigma+2)\theta\varphi z}{2}. \tag{4.19}$$

Additional inequalities follow upon application of the Laplace transform (1.9) to (5.5)–(5.8). We have

THEOREM 13.

$$\begin{aligned} &(1+\theta z)^{-\sigma} < {}_{p+1}F_p\left(\begin{matrix}\sigma, \alpha_p\\ \rho_p\end{matrix}\,\middle|\, -z\right) < 1 - \theta + \theta(1+z)^{-\sigma},\\ &z > 0, \qquad \sigma > 0, \qquad \rho_j \geqslant \alpha_j > 0, \qquad j = 1, 2,\ldots, p; \end{aligned} \tag{4.20}$$

$$\begin{aligned} &(1-\theta z)^{-\sigma} < {}_{p+1}F_p\left(\begin{matrix}\sigma, \alpha_p\\ \rho_p\end{matrix}\,\middle|\, z\right) < 1 - \theta + \theta(1-z)^{-\sigma},\\ &0 < z < 1, \qquad \sigma > 0, \qquad \rho_j \geqslant \alpha_j > 0, \qquad j = 1, 2,\ldots, p; \end{aligned} \tag{4.21}$$

$$\begin{aligned} &1 - \sigma\theta\left(1 - \frac{\varphi}{2}\right)z - \frac{\sigma\theta\varphi z}{2(1+z)^{\sigma+1}}\\ &\quad < {}_{p+1}F_p\left(\begin{matrix}\sigma, \alpha_p\\ \rho_p\end{matrix}\,\middle|\, -z\right) < 1 - \frac{\sigma\theta z}{(1+(\varphi z/2))^{\sigma+1}},\\ &z > 0, \qquad \sigma > 0, \qquad \rho_j \geqslant \alpha_j > 0, \qquad j = 1, 2,\ldots, p; \end{aligned} \tag{4.22}$$

$$\begin{aligned} &1 + \frac{\sigma\theta z}{(1-(\varphi z/2))^{\sigma+1}} < {}_{p+1}F_p\left(\begin{matrix}\sigma, \alpha_p\\ \rho_p\end{matrix}\,\middle|\, z\right)\\ &\quad < 1 + \sigma\theta\left(1 - \frac{\varphi}{2}\right)z + \frac{\sigma\theta\varphi z}{2(1-z)^{\sigma+1}},\\ &0 < z < 1, \qquad \sigma > 0, \qquad \rho_j \geqslant \alpha_j > 0, \qquad j = 1, 2,\ldots, p. \end{aligned} \tag{4.23}$$

Numerous other inequalities for ${}_{p+1}F_p$ can be found by enlarging upon the techniques and ideas enunciated. The theorems developed herein seem sufficient to indicate the general nature of expected results and we do not further pursue the subject. Similar type inequalities can be found hor hypergeometric functions of two or more variables, but we defer discussion on this point to a future paper.

V. Inequalities for Confluent Hypergeometric Functions, ${}_pF_p$ and a Particular G-Function

In Theorem 9, (4.12), replace z by z/a and let $a \to \infty$. Then by the confluence principle (see [13], or otherwise), we have

THEOREM 14.

$$-\frac{1}{\sigma} + \frac{(\sigma+1)}{\sigma}\left[1 + \frac{\theta z}{\sigma+1}\right]^{-1} < {}_{p+1}F_{p+1}\left(\begin{matrix}1, \alpha_p \\ \sigma, \rho_p\end{matrix}\,\middle|\, -z\right)$$
$$< 1 - \frac{\theta}{\varphi} + \frac{\theta}{\varphi}\left[1 + \frac{\varphi z}{\sigma}\right]^{-1}, \tag{5.1}$$
$$z > 0, \qquad \sigma > 0, \qquad \rho_j \geqslant \alpha_j > 0, \qquad j = 1, 2,..., p.$$

For ${}_1F_1$, we have the following inequalities:

THEOREM 15.

$$-1 + 2\,{}_2F_1\left(\begin{matrix}1, a \\ c\end{matrix}\,\middle|\, -\frac{z}{2}\right) < {}_1F_1\left(\begin{matrix}a \\ c\end{matrix}\,\middle|\, -z\right) < {}_2F_1\left(\begin{matrix}1, a \\ c\end{matrix}\,\middle|\, -z\right), \tag{5.2}$$

$$-1 + 2\left[1 + \frac{az}{2c}\right]^{-1} < {}_1F_1\left(\begin{matrix}a \\ c\end{matrix}\,\middle|\, -z\right)$$
$$< 1 - \frac{a(c+1)}{c(a+1)} + \frac{a(c+1)}{c(a+1)}\left[1 + \frac{(a+1)z}{c+1}\right]^{-1}, \tag{5.3}$$
$$z > 0, \qquad c \geqslant a > 0;$$

$$-\frac{1}{c} + \frac{(c+1)}{2}\left[1 + \frac{az}{c+1}\right]^{-1} < {}_1F_1\left(\begin{matrix}a \\ c\end{matrix}\,\middle|\, -z\right)$$
$$< \frac{1-a}{1+a} + \frac{2a}{a+1}\left[1 + \frac{(a+1)z}{2c}\right]^{-1}, \tag{5.4}$$
$$z > 0, \qquad a < 1, \qquad c > 0.$$

Proof. (5.2) follows from (3.7) with $n = 1$ and $\nu = 0$, using the beta transforms (1.6) and (1.7). (5.3) results from the coupling of (5.2) and (4.6). It is also a special case of (5.1) as is (5.4).

All ${}_pF_p$ inequalities become equalities if $z = 0$ or if any numerator parameter is zero.

Improved but very complicated approximations for ${}_pF_p$ can be obtained by using (3.7) and the beta transforms after the manner of the discussion surrounding (4.1)–(4.3). An attempt to get improved inequalities in this fashion leads to serious complications since $B_n(\nu, z)$ (see (3.3)) has non real zeros when $n > 1$. For simple and efficient approximations for ${}_pF_p$, see [12].

Some further easily proved inequalities for ${}_pF_p$ are given by

THEOREM 16.

$$e^{-\theta z} < {}_pF_p\left(\begin{matrix}\alpha_p\\ \rho_p\end{matrix}\middle| -z\right) < 1 - \theta + \theta e^{-z}, \tag{5.5}$$

$$e^{\theta z} < {}_pF_p\left(\begin{matrix}\alpha_p\\ \rho_p\end{matrix}\middle| z\right) < 1 - \theta + \theta e^{z}, \tag{5.6}$$

$$1 - \theta z\left(1 - \frac{\varphi}{2} + \frac{\varphi}{2}e^{-z}\right) < {}_pF_p\left(\begin{matrix}\alpha_p\\ \rho_p\end{matrix}\middle| -z\right) < 1 - \theta z e^{-\varphi z/2}, \tag{5.7}$$

$$\begin{gathered}1 + \theta z e^{\varphi z/2} < {}_pF_p\left(\begin{matrix}\alpha_p\\ \rho_p\end{matrix}\middle| z\right) < 1 + \theta z\left(1 - \frac{\varphi}{2} + \frac{\varphi}{2}e^{z}\right),\\ z > 0, \qquad \rho_j \geqslant \alpha_j > 0, \qquad j = 1, 2, \ldots, p.\end{gathered} \tag{5.8}$$

We now consider inequalities for a G-function of the form given by (1.10). See also (1.11)–(1.15) and (3.12). To this end, replace z by t in (4.10), multiply throughout by $t^{\epsilon-1}e^{-zt}\,dt$, integrate with respect to t from 0 to ∞ and apply (1.10) and (3.11). Then

$$\begin{aligned}&\left(\frac{\sigma\theta}{z}\right)^{-\epsilon} e^{z/\sigma\theta}\Gamma(1-\epsilon, z/\sigma\theta) < \frac{\Gamma(\rho_p)}{\Gamma(\sigma)\,\Gamma(\epsilon)\,\Gamma(\alpha_p)}\, G^{p+2,1}_{p+1,p+2}\left(z\,\middle|\,\begin{matrix}1, & \rho_p\\ \epsilon, \sigma, & \alpha_p\end{matrix}\right)\\ &\quad < 1 - \frac{2\sigma\theta}{(\sigma+1)\varphi} + \frac{2\sigma\theta}{(\sigma+1)\varphi}\left\{\frac{(\sigma+1)\varphi}{2z}\right\}^{-\epsilon} e^{2z/(\sigma+1)\varphi}\Gamma(1-\epsilon, 2z/(\sigma+1)\varphi),\\ &\quad z > 0, \qquad 0 < \sigma < 1, \qquad \epsilon > 0, \qquad \rho_j \geqslant \alpha_j > 0, \qquad j = 1, 2, \ldots, p.\end{aligned} \tag{5.9}$$

If $m = n = 1$, we have from (3.17):

$$1 - \frac{1-\nu}{z+1-\nu} < z^{1-\nu}e^{z}\Gamma(\nu, z) < 1 - \frac{1-\nu}{z+2-\nu}, \qquad z > 0, \quad \nu < 1. \tag{5.10}$$

Now combine the last two inequalities to get

THEOREM 17.

$$1-\frac{\sigma\epsilon\theta}{z+\sigma\epsilon\theta}<\frac{\Gamma(\rho_p)}{\Gamma(\sigma)\Gamma(\epsilon)\Gamma(\alpha_p)}G_{p+1,p+2}^{p+2,1}\left(z\,\middle|\,\begin{matrix}1,\rho_p\\ \sigma,\epsilon,\alpha_p\end{matrix}\right)$$
$$<1-\frac{\sigma\epsilon\theta}{z+[(\sigma+1)(\epsilon+1)\varphi/2]},$$
$$z>0,\qquad 0<\sigma\leqslant 1,\qquad \epsilon>0,\qquad \rho_j\geqslant\alpha_j>0,\qquad j=1,2,\ldots,p. \tag{5.11}$$

Inequalities for the G-function become equalities if $z\to\infty$, and likewise for certain values of the parameters as, for example, when $\nu=1$ in (5.10).

Improved inequalities and approximations for the G-functions in (5.11)–(5.17) can be obtained by using the discussion surrounding (4.1)–(4.3), and by using (3.17) for arbitrary m and n with $P_n(\nu, z)$ decomposed into a sum of partial fractions. Except for a multiplicative factor, $F_n(\nu, z)$, see (3.13), is the generalized Laguerre polynomial $L_n^{(\alpha)}(-z)$, $\alpha=1-a-\nu$; and if $\alpha>-1$, the zeros of $F_n(\nu, z)$ are simple and lie in the interval $-\infty<z<0$, see [11].

In a similar fashion, starting with (4.12), (4.15), (4.17) and (4.18), we get

THEOREM 18.

$$1-\frac{\epsilon a\theta/c}{z+[\epsilon(a+1)\theta/(c+1)]}<\frac{\Gamma(c)\,\Gamma(\rho_p)}{\Gamma(\epsilon)\,\Gamma(a)\,\Gamma(\alpha_p)}G_{p+2,p+3}^{p+3,1}\left(z\,\middle|\,\begin{matrix}1,c,\rho_p\\ \epsilon,1,a,\alpha_p\end{matrix}\right)$$
$$<1-\frac{\epsilon a\theta/c}{z+[(\epsilon+1)\alpha\varphi/c]}$$
$$z>0,\qquad 0<c\leqslant a,\qquad \epsilon>0,\qquad \rho_j\geqslant\alpha_j>0,\qquad j=1,2,\ldots,p; \tag{5.12}$$

$$1-\frac{\epsilon\theta}{(2-\sigma)}[z+\epsilon\theta]^{-1}+\frac{\epsilon(\sigma-1)^2\theta}{(2-\sigma)}[z+(\epsilon+1)(\sigma-1)\varphi]^{-1}$$
$$<\frac{\Gamma(\rho_p)}{\Gamma(\epsilon)\,\Gamma(\sigma)\,\Gamma(\alpha_p)}G_{p+1,p+2}^{p+2,1}\left(z\,\middle|\,\begin{matrix}1,\rho_p\\ \epsilon,\sigma,\alpha_p\end{matrix}\right)$$
$$<1+\frac{\epsilon\sigma(\sigma-1)\theta}{(2-\sigma)}\left[z+\frac{\epsilon\sigma\theta}{2}\right]^{-1}-\frac{\epsilon\sigma\theta}{(2-\sigma)}[z+(\epsilon+1)\varphi]^{-1},$$
$$z>0,\qquad 1\leqslant\sigma\leqslant 2,\qquad \epsilon>0,\qquad \rho_j\geqslant\alpha_j>0,\qquad j=1,2,\ldots,p; \tag{5.13}$$

$$1 - \epsilon\sigma\theta[z + (\epsilon + 1)(\sigma + 1)\varphi]^{-1} < \frac{\Gamma(\rho_p)}{\Gamma(\epsilon)\,\Gamma(\sigma)\,\Gamma(\alpha_p)}\, G^{p+2,1}_{p+1,p+2}\left(z \,\middle|\, \begin{matrix} 1, \rho_p \\ \epsilon, \sigma, \alpha_p \end{matrix}\right)$$
$$< 1 - \frac{\epsilon\sigma\theta}{(\sigma + 2)z} - \frac{\epsilon\sigma(\sigma + 1)\theta}{\sigma + 2}\left[z + \epsilon(\sigma + 2)\frac{\theta}{2}\right]^{-1},$$
$$z > 0, \qquad -1 < \sigma < 0, \qquad \epsilon > 0, \qquad \rho_j \geqslant \alpha_j > 0, \qquad j = 1, 2,\ldots, p; \tag{5.14}$$

$$1 - \epsilon\sigma\theta\left[z + \frac{(\epsilon + 1)(\sigma + 1)\varphi}{2}\right]^{-1} < \frac{\Gamma(\rho_p)}{\Gamma(\epsilon)\,\Gamma(\sigma)\,\Gamma(\alpha_p)}\, G^{p+2,1}_{p+1,p+2}\left(z \,\middle|\, \begin{matrix} 1, \rho_p \\ \epsilon, \sigma, \alpha_p \end{matrix}\right)$$
$$< 1 - \frac{\epsilon\sigma\theta}{z} + \frac{3\epsilon\sigma(\sigma + 1)\theta\varphi}{2(\sigma + 2)\eta z} - \frac{3\epsilon\sigma(\sigma + 1)\theta\varphi}{2(\sigma + 2)\eta}\left[z + \frac{\epsilon(\sigma + 2)\eta}{3}\right]^{-1},$$
$$z > 0, \qquad -1 < \sigma < 0, \qquad \epsilon > 0, \qquad \rho_j \geqslant \alpha_j > 0, \qquad j = 1, 2,\ldots, p. \tag{5.15}$$

It is readily shown that the left-hand side of (5.14) is less than the left-hand side of (5.15), and that the right-hand side of (5.15) is less than the right-hand side of (5.14) provided

$$\theta z + \epsilon(\sigma + 2)\,\theta\eta/3 < \varphi z + \epsilon(\sigma + 2)\,\theta\varphi/2. \tag{5.16}$$

Using (4.20) and (4.22), each with (1.10), we get

THEOREM 19.

$$\frac{1}{\Gamma(\epsilon)\,\Gamma(\sigma)}\, G^{2,1}_{1,2}\left(\frac{z}{\theta} \,\middle|\, \begin{matrix} 1 \\ \epsilon, \sigma \end{matrix}\right) < \frac{\Gamma(\rho_p)}{\Gamma(\epsilon)\,\Gamma(\sigma)\,\Gamma(\alpha_p)}\, G^{p+2,1}_{p+1,p+2}\left(z \,\middle|\, \begin{matrix} 1, \rho_p \\ \epsilon, \sigma, \alpha_p \end{matrix}\right)$$
$$< 1 - \theta + \frac{\theta}{\Gamma(\epsilon)\,\Gamma(\sigma)}\, G^{2,1}_{1,2}\left(z \,\middle|\, \begin{matrix} 1 \\ \epsilon, \sigma \end{matrix}\right),$$
$$z > 0, \qquad \sigma > 0, \qquad \epsilon > 0, \qquad \rho_j \geqslant \alpha_j > 0, \qquad j = 1, 2,\ldots, p; \tag{5.17}$$

$$1 - \frac{\epsilon\sigma\theta(1 - \varphi/2)}{z} - \frac{\theta\varphi}{2\Gamma(\epsilon)\,\Gamma(\sigma)z}\, G^{2,1}_{1,2}\left(z \,\middle|\, \begin{matrix} 1 \\ \epsilon + 1, \sigma + 1 \end{matrix}\right)$$
$$< \frac{\Gamma(\rho_p)}{\Gamma(\epsilon)\,\Gamma(\sigma)\,\Gamma(\alpha_p)}\, G^{p+2,1}_{p+1,p+2}\left(z \,\middle|\, \begin{matrix} 1, \rho_p \\ \epsilon, \sigma, \alpha_p \end{matrix}\right)$$
$$< 1 - \frac{\theta}{\Gamma(\epsilon)\,\Gamma(\sigma)z}\, G^{2,1}_{1,2}\left(\frac{\varphi z}{2} \,\middle|\, \begin{matrix} 1 \\ \epsilon + 1, \sigma + 1 \end{matrix}\right),$$
$$z > 0, \qquad \sigma > 0, \qquad \epsilon > 0, \qquad \rho_j \geqslant \alpha_j > 0, \qquad j = 1, 2,\ldots, p. \tag{5.18}$$

VI. Examples

1. Consider

$$x^{-1} \arctan x = {}_2F_1\left(\begin{matrix}1, \frac{1}{2} \\ \frac{3}{2}\end{matrix} \middle| -x^2\right). \tag{6.1}$$

Apply (4.6). Then

$$\left(1 + \frac{x^2}{3}\right)^{-1} < x^{-1} \arctan x < \frac{4}{9} + \frac{5}{9}\left[1 + \frac{3x^2}{5}\right]^{-1}, \qquad x > 0, \tag{6.2}$$

whence with $x = 1$, $3 < \pi < 19/6$. Integrate (6.2) from 0 to x and make use of (6.2). Then

$$\begin{aligned} L = \left(1 + \frac{x^2}{9}\right)^{-1} &< x^{-1} \int_0^x t^{-1} \arctan t \, dt < R \\ &= \frac{56}{81} + \frac{25}{81}\left[1 + \frac{9x^2}{25}\right]^{-1}, \qquad x > 0. \end{aligned} \tag{6.3}$$

In particular if $x = 1$, the integral is Catalan's constant which to five decimal places equals 0.91597. For $x = 1$, $L = 0.9$, and $R = 2529/2754 = 0.91830$.

2. Similarly, for

$$x^{-1} \ln(1 + x) = {}_2F_1\left(\begin{matrix}1, 1 \\ 2\end{matrix} \middle| -x\right), \tag{6.4}$$

we have

$$\left[1 + \frac{x}{2}\right]^{-1} < x^{-1} \ln(1 + x) < \frac{1}{4} + \frac{3}{4}\left[1 + \frac{2x}{3}\right]^{-1}, \qquad x > 0, \tag{6.5}$$

and so $2/3 < \ln 2 < 0.7$. Also

$$\begin{aligned} L = \left[1 + \frac{x}{4}\right]^{-1} &< x^{-1} \int_0^x t^{-1} \ln(1 + t) \, dt < R \\ &= \frac{1}{4} + \frac{3}{4}\left(1 + \frac{x}{9}\right)\left[1 + \frac{4x}{9}\right]^{-1}, \qquad x > 0. \end{aligned} \tag{6.6}$$

If $x = 1$, the integral is $\pi^2/12 = 0.82247$, $L = 0.8$, and $R = 43/52 = 0.82692$.

3. Let

$$F(z) = {}_2F_1\left(\begin{matrix}\frac{1}{2}, \frac{3}{2} \\ \frac{5}{2}\end{matrix} \middle| -z\right) = (1 + z)^{1/2} \, {}_2F_1\left(\begin{matrix}1, 2 \\ \frac{5}{2}\end{matrix} \middle| -z\right). \tag{6.7}$$

Appropriate use of (4.7), (4.12) and (4.15) for the first form of $F(z)$ yields the respective inequalities, all for $z > 0$:

$$\left[1+\frac{3z}{10}\right]^{-1} < F(z) < \frac{11}{25}+\frac{14}{25}\left[1+\frac{15z}{28}\right]^{-1},$$

$$-\frac{1}{5}+\frac{6}{5}\left[1+\frac{z}{4}\right]^{-1} < F(z) < \frac{8}{15}+\frac{7}{15}\left[1+\frac{9z}{14}\right]^{-1},$$

$$2\left[1+\frac{z}{5}\right]^{-1}-1+\frac{7}{15}\left[1-\left\{1+\frac{3z}{14}\right\}^{-1}\right] < F(z) < 3-2\left[1+\frac{3z}{20}\right]^{-1} - \frac{7}{5}\left[1-\left\{1+\frac{3z}{7}\right\}^{-1}\right]. \tag{6.8}$$

If $z = \frac{1}{2}$, we have $F = F(\frac{1}{2}) = 0.88055$ and from (6.8) we obtain, respectively,

$$\begin{aligned} 0.86957 &< F < 0.88169, \\ 0.86667 &< F < 0.88649, \\ 0.86334 &< F < 0.89248. \end{aligned} \tag{6.9}$$

Now apply (4.6) to the second form of $F(z)$. Then

$$(1+z)^{1/2}\left[1+\frac{4z}{5}\right] < F(z) < \frac{(1+z)^{1/2}}{15}\left[1+14\left\{1+\frac{6z}{7}\right\}^{-1}\right] \tag{6.10}$$

and, for $z = \frac{1}{2}$,

$$0.87482 < F(z) < 0.88182. \tag{6.11}$$

4. Let

$$G(z) = {}_2F_1\left(\begin{matrix}-\frac{1}{2}, \frac{1}{2} \\ \frac{3}{2}\end{matrix}\middle|\, -z\right). \tag{6.12}$$

If $z = \frac{1}{2}$, $G = G(\frac{1}{2}) = 1.07799$ and from (4.17) and (4.18), respectively, we have

$$\begin{aligned} 1.07246 &< G < 1.08025, \\ 1.07752 &< G < 1.07803. \end{aligned} \tag{6.13}$$

5. Consider the complete elliptic integral of the first kind

$$\mathbf{K}(k) = \frac{\pi}{2}\,{}_2F_1\left(\begin{matrix}\frac{1}{2}, \frac{1}{2} \\ 1\end{matrix}\middle|\, k^2\right) = \frac{\pi}{2}(1-k^2)^{-1/2}\,{}_2F_1\left(\begin{matrix}\frac{1}{2}, \frac{1}{2} \\ 1\end{matrix}\middle|\, -\frac{k^2}{1-k^2}\right). \tag{6.14}$$

From (4.7) we have

$$\frac{2\pi(1-k^2)^{1/2}}{4-3k^2} < \mathbf{K}(k) < \frac{\pi(16-11k^2)(1-k^2)^{-1/2}}{2(16-7k^2)}, \qquad 0<k<1. \tag{6.15}$$

The complete elliptic integral of the second kind is

$$\mathbf{E}(k) = \frac{\pi}{2}\, {}_2F_1\left(\begin{matrix}-\frac{1}{2}, \frac{1}{2}\\ 1\end{matrix}\,\middle|\, k^2\right) = \frac{\pi}{2}(1-k^2)^{1/2}\, {}_2F_1\left(\begin{matrix}-\frac{1}{2}, \frac{1}{2}\\ 1\end{matrix}\,\middle|\, -\frac{k^2}{1-k^2}\right), \quad (6.16)$$

and

$$\frac{\pi}{2}(1-k^2)^{1/2}\left[1+\frac{4k^2}{16-13k^2}\right] < \mathbf{E}(k)$$
$$< \frac{\pi}{2}(1-k^2)^{1/2}\left[1+\frac{k^2(48-37k^2)}{16(1-k^2)(12-7k^2)}\right], \qquad 0<k<1. \quad (6.17)$$

From (4.21) we have

$$\frac{\pi}{2}\left(\frac{2}{2-k^2}\right)^{1/2} < \mathbf{K}(k) < \frac{\pi}{4}[1+(1-k^2)^{-1/2}], \qquad 0<k<1, \quad (6.18)$$

and from (4.21) and (2.8) with $r = 1$, we get

$$\frac{\pi}{2}\left[1-\frac{k^2}{32}\{7+(1-k^2)^{-3/2}\}\right] < \mathbf{E}(k) < \frac{\pi}{2}\left[1-\frac{k^2}{4}\left(1-\frac{k^2}{8}\right)^{-3/2}\right],$$
$$0<k<1. \quad (6.19)$$

Improved inequalities for the above complete elliptic integrals can be obtained by first applying a Landen type transformation. Extensive approximations for the three kinds of complete and incomplete elliptic integrals based on the Padé approximations for the square root have been given in my recent paper [14].

6. If we apply the Kummer formulas (1.17) and (1.18) to (5.2), then with an appropriate change of notation we get

$$-1+4(z+2)^{-1}\, {}_2F_1\left(\begin{matrix}1, a\\ c\end{matrix}\,\middle|\, \frac{z}{z+2}\right) < e^{-z}\, {}_1F_1\left(\begin{matrix}a\\ c\end{matrix}\,\middle|\, z\right)$$
$$< (1+z)^{-1}\, {}_2F_1\left(\begin{matrix}1, a\\ c\end{matrix}\,\middle|\, \frac{z}{z+1}\right), \qquad z>0, \quad c>a>0. \quad (6.20)$$

Under the latter hypotheses, the ${}_2F_1$ on the right of (6.20) is less than the same ${}_2F_1$ with argument unity which can be summed provided $c > a + 1$, see [15]. Thus

$$(1+z)\, e^{-z}\, {}_1F_1\left(\begin{matrix}a\\ c\end{matrix}\,\middle|\, z\right) < \frac{(c-1)}{c-1-a}, \qquad z>0, \quad c-1>a>0. \quad (6.21)$$

7. The modified Bessel function of the first kind can be defined by

$$I_\nu(z) = \frac{(z/2)^\nu e^z}{\Gamma(\nu+1)}\,{}_1F_1(\nu+\tfrac{1}{2};\, 2\nu+1;\, -2z). \tag{6.22}$$

Application of (5.3) gives

$$\frac{1-z/2}{1+z/2} < \Gamma(\nu+1)(2/z)^\nu\, e^{-z} I_\nu(z) < \frac{1}{2\nu+3} + \frac{2(\nu+1)}{2\nu+3}\left[1+\frac{(2\nu+3)z}{2(\nu+1)}\right]^{-1},$$
$$z > 0, \quad \nu \geqslant -\tfrac{1}{2}. \tag{6.23}$$

The left-hand inequality is very weak unless z is quite small. From (5.4), we have

$$\left[1-\frac{z}{2(\nu+1)}\right]\left[1+\frac{(2\nu+1)z}{2(\nu+1)}\right]^{-1} < \Gamma(\nu+1)(2/z)^\nu\, e^{-z} I_\nu(z)$$
$$< \frac{1-2\nu}{2\nu+3} + \frac{2(2\nu+1)}{2\nu+3}\left[1+\frac{(2\nu+3)z}{2(2\nu+1)}\right]^{-1}, \qquad z>0, \quad -\tfrac{1}{2} \leqslant \nu \leqslant \tfrac{1}{2}. \tag{6.24}$$

If $\nu = 0$, (6.23), and (6.24) coincide. Finally, from (5.5) we get

$$e^{-z} < \Gamma(\nu+1)(2/z)^\nu\, e^{-z} I_\nu(z) < \tfrac{1}{2}(1+e^{-2z}), \qquad z>0, \quad \nu > -\tfrac{1}{2}. \tag{6.25}$$

8. The modified Bessel function of the second kind can be expressed in either of the forms

$$K_\nu(z) = \pi^{1/2} e^{-z}(2z)^\nu\, \psi(\tfrac{1}{2}+\nu;\, 1+2\nu;\, 2z), \tag{6.26}$$

or

$$K_\nu(z) = \frac{(\pi/2z)^{1/2}\, e^{-z}}{\Gamma(\frac{1}{2}+\nu)\,\Gamma(\frac{1}{2}-\nu)}\, G^{2,1}_{1,2}\left(2z \,\middle|\, \begin{matrix} 1 \\ \frac{1}{2}+\nu,\ \frac{1}{2}-\nu \end{matrix}\right). \tag{6.27}$$

From (5.11) and (5.15), we get the respective inequalities

$$1 - \frac{\frac{1}{2}(\frac{1}{4}-\nu^2)}{z+\frac{1}{2}(\frac{1}{4}-\nu^2)} < (2z/\pi)^{1/2}\, e^z K_\nu(z) < 1 - \frac{\frac{1}{2}(\frac{1}{4}-\nu^2)}{z+\frac{1}{4}(9/4-\nu^2)},$$
$$z > 0, \qquad 0 \leqslant \nu < \tfrac{1}{2}; \tag{6.28}$$

$$1 + \frac{\frac{1}{2}(\nu^2-\frac{1}{4})}{z+\frac{1}{4}(9/4-\nu^2)} < (2z/\pi)^{1/2}\, e^z K_\nu(z) < 1 + \frac{(\nu^2-\frac{1}{4})(\frac{1}{2}+\nu)}{4z(5/2-\nu)}$$
$$+ \frac{3(\nu^2-\frac{1}{4})(3/2-\nu)}{4(5/2-\nu)[z+((\frac{1}{2}+\nu)(5/2-\nu)/6)]}, \qquad z>0, \quad \tfrac{1}{2} < \nu < 3/2. \tag{6.29}$$

Notice that (6.28) and (6.29) become equalities as $z \to \infty$ or if $\nu = \frac{1}{2}$. Also (6.29) becomes an equality if $\nu = 3/2$.

Taking $\nu = 0$, we have

$$L(z) = \frac{8z}{8z+1} < F(z) = \left(\frac{2z}{\pi}\right)^{1/2} e^z K_0(z) < R(z) = \frac{16z+7}{16z+9}, \qquad z > 0. \tag{6.30}$$

The utility of these inequalities is made manifest by the following table:

z	$L(z)$	$F(z)$	$R(z)$
0.01	0.07407	0.38049	0.78166
0.10	0.44444	0.67679	0.81132
0.50	0.80000	9.85989	0.88235
1.0	0.88889	0.91315	0.92000
2.0	0.94118	0.94961	0.95122
4.0	0.96970	0.97230	0.97260
10.0	0.98765	0.98814	0.98817

(6.31)

Notice that for $z \geqslant \frac{1}{2}$, the arithmetic mean of $L(z)$ and $R(z)$ approximates $F(z)$ to within about 2.2%. This is quite remarkable as $K_0(z)$ has a logarithmic singularity at $z = 0$.

9. The parabolic cylinder function is given by

$$D_\nu(z) = 2^{(\nu-1)/2} e^{-(z^2/4)} z \psi\left(\frac{1-\nu}{2}, \frac{3}{2}; \frac{z^2}{2}\right) \tag{6.32}$$

or

$$D_\nu(z) = \frac{2z^\nu e^{-(z^2/4)}}{\pi^{1/2}\Gamma(-\nu/2)} G_{1,2}^{2,1}\left(\frac{z^2}{2} \,\middle|\, \begin{matrix} 1 \\ \frac{1-\nu}{2}, -\frac{\nu}{2} \end{matrix}\right). \tag{6.33}$$

From (5.11) and (5.15) we have the respective inequalities

$$1 + \frac{\frac{1}{2}\nu(1-\nu)}{z^2 - \frac{1}{2}\nu(1-\nu)} < z^{-\nu} e^{z^2/4} D_\nu(z) < 1 + \frac{\frac{1}{2}\nu(1-\nu)}{z^2 + \frac{1}{4}(3-\nu)(2-\nu)}, \qquad z > 0, \quad -2 \leqslant \nu < 0; \tag{6.34}$$

$$1 + \frac{\frac{1}{2}\nu(1-\nu)}{z^2 + \frac{1}{4}(3-\nu)(2-\nu)} < z^{-\nu} e^{z^2/4} D_\nu(z) < 1 + \frac{\nu(1-\nu)(\nu+2)}{4(4-\nu)z^2} + \frac{3\nu(1-\nu)(2-\nu)}{4(4-\nu)[z^2 + ((1-\nu)(4-\nu)/6)]}, \qquad z > 0, \quad 0 < \nu < 1. \tag{6.35}$$

Both (6.34) and (6.35) become equalities as $z \to \infty$ or as $\nu \to 0$. Also (6.35) becomes an equality if $\nu = 1$.

VII. Other Inequalities

Inequalities for the special functions appear infrequently in the literature. Gautschi [16] (see also the references given there) has developed a two-sided inequality for the incomplete gamma function $\Gamma(\nu, z)$. More recently, Carlson [17] has developed two-sided inequalities for a hypergeometric function of n-variables which includes ${}_{p+1}F_p$ as a special case. Some of these inequalities are closely related to those presented here. Application of the confluence principle to one of his inequalities for ${}_2F_1$ leads to (5.5) and (5.6) with $p = 1$. Neither of the above authors makes use of transforms to develop inequalities for other special functions. In a future paper, we intend to investigate this aspect of the subject, and to apply our techniques to develop inequalities for hypergeometric functions of several variables.

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